Circumventing the relative degree condition in sliding mode design

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Summary. Classical sliding mode design approaches assume the transfer function matrix between the driving signal and the measured output of interest must be minimum phase and relative degree one. For the control case, the driving signal will be the control input and for the observer problem the driving signal is likely to be an unknown input which the observer seeks to reconstruct. This chapter demonstrates that the relative degree condition can be weakened if the nominal linear system used for the controller or observer design is combined with sliding mode exact differentiators to essentially generate additional independent output signals from the available measurements. It is shown that the transmission zeros of the original plant appear directly in the reduced order sliding mode dynamics relating to the augmented system in both cases. In the case of output feedback sliding mode control design for MIMO systems of any relative degree, a super twisting control algorithm is shown to provide robust control performance. Nonlinear simulation results for a ninth order nonlinear description of a web transport system, which does not satisfy the usual relative degree one condition, are used to demonstrate both the control design approach and the design of an unknown input observer.

Key words: sliding mode; sliding mode differentiators; output feedback; observer design

1 Introduction

A continuous time sliding mode is generated by means of discontinuities in the applied injection signals, about a surface in the state space [17, 33, 40]. The discontinuity surface (usually known as the sliding surface) is attained from any initial condition ideally in a finite time interval. Provided the injection signals are designed appropriately, the motion when constrained to the surface (the sliding mode) is completely insensitive to so-called matched uncertainties, i.e. uncertainties that lie within the range space of the matrix distributing the injection signals. Much early work in this area related to control problems and assumed all of the states were available for use both in the switching function evaluation and also by the control law. For practical application however, the case when only limited state information is available is of interest.

A number of algorithms have been developed for robust stabilization of uncertain systems which are based on sliding surfaces and output feedback control schemes [16], [45]. In [45] a geometric condition is developed to guarantee the existence of the sliding surface and the stability of the reduced order sliding motion. Edwards and Spurgeon derived an algorithm [16], [17] which is convenient for practical use. In both these results, it is required that the disturbance considered is matched, i.e. acts in the channels of the inputs. In many cases, however, the disturbance suffered by practical systems does not act in the input channel. Unlike the matched case, any mismatched disturbance impinges on the sliding mode dynamics and affects the behaviour of the sliding mode directly [44]. Based on the work in [45], some dynamic output feedback control schemes have been proposed [30], [35]. Unfortunately, in all the above output feedback sliding mode control schemes, it is an *a priori* requirement that the system under consideration is minimum phase and relative degree one.

The concept of sliding mode control has been extended to the problem of state estimation by an observer, for linear systems [40], uncertain linear systems [15, 42] and nonlinear systems [1, 13, 36]. Using the same design principles as for variable structure control, the observer trajectories are constrained to evolve after a finite time on a suitable sliding manifold by the use of a discontinuous output injection signal (the sliding manifold is usually given by the difference between the observer and the system output). Subsequently the sliding motion provides an estimate (asymptotically or in finite time) of the system states. Sliding mode observers have been shown to be efficient in many applications, such as in robotics [4, 27], electrical engineering [11, 21, 41], and fault detection [20, 22]. The necessary and sufficient conditions for the existence of a 'classical' sliding mode observer[3] as described in [15, 42] is that the transfer function matrix between the unmeasurable inputs (or disturbances) and the measured outputs must be minimum phase and relative degree one.

This chapter shows how it is possible to broaden the class of systems for which both sliding mode output feedback controllers and observers can be designed. It is shown that the relative degree condition can be weakened in both cases if the sliding mode controller or observer is combined with sliding mode exact differentiators to generate additional independent output signals from the available measurements. The work has its roots in the contribution of [5] where an output feedback sliding mode controller for MIMO systems of any relative degree is considered. It is assumed that the input explicitly appears first in the r-th time derivatives of each of the p outputs of the system. In [5], the authors take r derivatives of each measured output and introduce an integral sign function control, whereby effectively the control is designed to

[3] A precise observer description will be given later in the Chapter.

determine the derivative of the actual applied input signal. Here the number of outputs requiring differentiation is minimized and a robust sliding mode differentiator is presented as the means to construct the extended output signal. It is shown that the transmission zeros of the triple used to design the controller or observer appear directly in the reduced order sliding mode dynamics. For the static output feedback control problem, a twisting control algorithm is shown to provide robust control performance. For the sliding mode observer design problem, a classical first order observer is described which estimates the system states and any unknown inputs. Nonlinear simulation results for a ninth order nonlinear description of a web-transport system, which does not satisfy the usual relative degree conditions required for sliding mode output feedback controller or observer design, are used to demonstrate the efficacy of the approach.

2 Motivation and General Problem Statement

Consider an uncertain dynamical system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + f(t, x, u) \tag{1}$$

$$y = \begin{bmatrix} y_1 & \cdots & y_p \end{bmatrix}^T = Cx, \ y_i = C_i x \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ with $m \leq p < n$. It is assumed that the system (1) is Bounded Input Bounded State (BIBS), that the nominal linear system (A, B, C) is known with (A, B) controllable and that the input and output matrices B and C are both full rank. The function $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ represents system nonlinearities, uncertainties, disturbances or any other unknown input present in the system. It is assumed to be bounded as well as having a bounded first time derivative: i.e. there exists a smooth vector field $\xi(t, x, u) \in \mathbb{R}^q$, a known constant matrix $D \in \mathbb{R}^{n \times q}$ and some known constants K and $K^{'}$ such that:

$$f(t, x, u) = D\xi(t, x, u), \quad \|\xi(t, x, u)\| < K, \quad \left\|\dot{\xi}(t, x, u)\right\| < K^{'}$$

In the case of the development of a control law based on output measurements only, all uncertainties have been assumed to be matched, so that $D = B$ (and as a consequence, $q = m$). Then, the problem is to induce an ideal sliding motion on the surface

$$S = \{x \in \mathbb{R}^n : FCx = 0\} \tag{3}$$

for some selected matrix $F \in \mathbb{R}^{m \times p}$. It is well known that for a unique equivalent control to exist, the matrix $FCB \in \mathbb{R}^{m \times m}$ must have full rank. As

$$\text{rank}(FCB) \leq min\{\text{rank}(F), \text{rank}(CB)\} \tag{4}$$

it follows that both F and CB must have full rank. As F is a design parameter, it can be chosen to be full rank. A necessary condition for FCB to be full rank, and thus for solvability of the output feedback sliding mode design problem, thus becomes that CB must have rank m. If this rank condition holds and any invariant zeros of the triple (A, B, C) lie in $\mathbb{C}_-$, then the existence of a matrix F defining the surface (3) which provides a stable sliding motion with a unique equivalent control is determined from the stabilizability by output feedback of a specific, well-defined subsystem of the plant [17]. Here, the first aim is to extend the existing results so that a sliding mode controller based on output measurements can be designed for the system (1-2) when rank(CB) is strictly less than m.

The second aim is to develop a sliding mode observer for the system (1), when driven by unknown inputs ξ. Without loss of generality, it can be assumed that rank$(D) = q$. Consider a sliding mode observer of the general form

$$\dot{\hat{x}} = A\hat{x} + Bu + G_l(y - C\hat{x}) + G_n v_c \tag{5}$$

where G_l and G_n are design gains and v_c is an injection signal which depends on the output estimation error in such a way that a sliding motion in the state estimation error space is induced in finite time. The objective is to ensure the state estimation error $e = x - \hat{x}$ is asymptotically stable and independent of the unknown signal ξ during the sliding motion. As argued in [17] necessary and sufficient conditions to solve this problem are: the invariant zeros of $\{A, D, C\}$ lie in $\mathbb{C}_-$ and

$$\text{rank}(CD) = \text{rank}(D) = q. \tag{6}$$

In this chapter, the existing results are extended so that a sliding mode observer based on output measurements can be designed for the system (1-2) when rank(CD) is strictly less than q.

The next section will explore an output extension approach to circumvent the relative degree condition for both sliding mode controller and observer problems.

3 Generation of the extended output

Introduce the notion of *relative degree* $\mu_j \in \mathbb{N}^*$, $1 \le j \le p$ of the system with respect to the output y_j, that is to say the number of times the output y_j must be differentiated in order to have the unknown input ξ explicitly appear. Thus, μ_j is defined as follows:

$$C_j A^k D = 0, \text{ for all } k < \mu_j - 1$$
$$C_j A^{\mu_j - 1} D \neq 0.$$

Without loss of generality, it is assumed that $\mu_1 \le ... \le \mu_p$.
The following assumptions are made:

- the invariant zeros of $\{A, D, C\}$ lie in $\mathbb{C}_-$
- there exists a full rank matrix

$$\tilde{C} = \begin{bmatrix} C_1 \\ \vdots \\ C_1 A^{\mu_{\alpha_1}-1} \\ \vdots \\ C_p \\ \vdots \\ C_p A^{\mu_{\alpha_p}-1} \end{bmatrix} \tag{7}$$

where the integers $1 \leq \mu_{\alpha_i} \leq \mu_i$ are such that $\text{rank}(\tilde{C}D) = \text{rank}(D)$ and the μ_{α_i} are chosen such that $\sum_{i=1}^{p} \mu_{\alpha_i} \triangleq \tilde{p}$ is minimal.

The following lemma will demonstrate that the invariant zeros of the triple $\{A, D, C\}$ and the newly created triple with additional (derivative) outputs $\{A, D, \tilde{C}\}$ are identical.

Lemma 1. *The invariant zeros of the triples* $\{A, D, C\}$ *and* $\{A, D, \tilde{C}\}$ *are identical.*

Proof: Suppose $s_0 \in \mathbb{C}$ is an invariant zero of $\{A, D, \tilde{C}\}$. Consequently $\tilde{P}(s)|_{s=s_0}$ loses normal rank, where $\tilde{P}(s)$ is Rosenbrock's system matrix defined by

$$\tilde{P}(s) := \begin{bmatrix} sI - A & D \\ \tilde{C} & 0 \end{bmatrix}$$

Since by assumption $p \geq m$, this implies $\tilde{P}(s)$ loses column rank and therefore there exist non-zero vectors η_1 and η_2 such that

$$\begin{aligned} (s_0 I - A)\eta_1 + D\eta_2 &= 0 \\ \tilde{C}\eta_1 &= 0 \end{aligned}$$

From the definition of $\tilde{C}$, $\tilde{C}\eta_1 = 0 \Rightarrow C\eta_1 = 0$. Consequently

$$\begin{aligned} (s_0 I - A)\eta_1 + D\eta_2 &= 0 \\ C\eta_1 &= 0 \end{aligned}$$

and so $P(s)|_{s=s_0}$ loses column rank where

$$P(s) := \begin{bmatrix} sI - A & D \\ C & 0 \end{bmatrix}$$

is Rosenbrock's System Matrix for the triple $\{A, D, C\}$. Therefore any invariant zero of $\{A, D, \tilde{C}\}$ is an invariant zero of $\{A, D, C\}$.

Now suppose $s_0 \in \mathbb{C}$ is an invariant zero of $\{A, D, C\}$. This implies the existence of non-zero vectors η_1 and η_2 such that

$$(s_0 I - A)\eta_1 + D\eta_2 = 0 \tag{8}$$
$$C\eta_1 = 0 \tag{9}$$

The first (sub) equation of (9) implies $C_1\eta_1 = 0$. Suppose $\mu_{\alpha_1} > 1$. Then multiplying (8) by C_1 gives

$$s_0 \underbrace{C_1\eta_1}_{=0} - C_1 A\eta_1 + \underbrace{C_1 D}_{=0}\eta_2 = 0$$

which implies $C_1 A\eta_1 = 0$. By an inductive argument it follows that $C_1 A^k \eta_1 = 0$ for $k \le \mu_{\alpha_1} - 1$. Repeating this analysis for C_2 up to C_p it follows

$$C_j A^k \eta_1 = 0 \qquad \text{for } k \le \mu_{\alpha_j} - 1, \quad j = 1 \ldots p$$

and therefore

$$\tilde{C}\eta_1 = 0 \tag{10}$$

Consequently, from (10) and (8), s_0 is an invariant zero of the triple $\{A, D, \tilde{C}\}$ and the lemma is proved. ♯

Implementation of the differentiators required to construct the extended output signals will now be discussed. The aim is to recover in finite time knowledge of the partially measured output vector generated by $\tilde{C}x$. The problem can be seen as one of designing an observer for a system that can be put in a so-called canonical triangular observable form. Most of the sliding mode observer designs for such a form are based on a step-by-step procedure using successive filtered values of the so-called equivalent output injections obtained from recursive first order sliding mode observers (see e.g. [1, 12, 13, 25, 32, 43]). However, the approximation of the equivalent injections by low pass filters at each step will typically introduce some delays that lead to inaccurate estimates or to instability for high order systems. To overcome this problem, the discontinuous first order sliding mode output injection is replaced by a continuous second order sliding mode one.

Define the following sliding mode observer based on a so-called step-by-step observer:

$$\begin{cases} \dot{y}_i^1 = \nu\left(y_i - y_i^1\right) + C_i Bu \\ \dot{y}_i^2 = E_1 \nu\left(\tilde{y}_i^2 - y_i^2\right) + C_i ABu \\ \vdots \\ \dot{y}_i^{\mu_{\alpha_i}-1} = E_{\mu_{\alpha_i}-2}\nu\left(\tilde{y}_i^{\mu_{\alpha_i}-1} - y_i^{\mu_{\alpha_i}-1}\right) + C_i A^{\mu_{\alpha_i}-2} Bu \end{cases}$$

for $1 \le i \le p$, with

$$\begin{aligned} \tilde{y}_i^1 &:= y_i \\ \tilde{y}_i^j &:= \nu\left(\tilde{y}_i^{j-1} - y_i^{j-1}\right),\ 2 \le j \le \mu_{\alpha_i} - 1 \end{aligned}$$

where the continuous output error injection $\nu(\cdot)$ is given by the so-called super twisting algorithm [31]:

$$\begin{cases} \nu(s) = \varphi(s) + \lambda_s \left|s\right|^{\frac{1}{2}} \operatorname{sign}(s) \\ \dot{\varphi}(s) = \alpha_s \operatorname{sign}(s) \\ \lambda_s, \alpha_s > 0 \end{cases} . \tag{11}$$

For $j = 1, ..., \mu_{\alpha_i} - 2$, the scalar functions E_i are defined as

$$E_j = 1 \text{ if } \left|\tilde{y}_i^k - y_i^k\right| \leq \varepsilon, \text{ for all } k \leq j \text{ else } E_i = 0$$

where ε is a small positive constant. This is an anti-peaking structure [37]. As argued in [1], with this particular function, the manifolds are reached one by one. At each step, a sub-dynamic of dimension one is obtained and consequently no peaking phenomena appear. Define the augmented output estimation error $e_y = \tilde{C}x - \bar{y}$, with

$$e_y \triangleq \left[e_1^1, ..., e_1^{\mu_{\alpha_1}-1}, ..., e_p^1, ..., e_p^{\mu_{\alpha_p}-1}\right]^T \tag{12}$$

$$\bar{y} = \left[y_1^1, ..., y_1^{\mu_{\alpha_1}-1}, ..., y_p^1, ..., y_p^{\mu_{\alpha_p}-1}\right]^T \tag{13}$$

then it is straightforward to show that:

$$\begin{cases} \dot{e}_i^1 = C_i\left(Ax + Bu + D\xi\right) - \nu\left(y_i - y_i^1\right) - C_i Bu = C_i Ax - \nu\left(y_i - y_i^1\right) \\ \dot{e}_i^2 = C_i A^2 x - E_1 \nu\left(\tilde{y}_i^2 - y_i^2\right) \\ \vdots \\ \dot{e}_i^{\mu_{\alpha_i}-1} = C_i A^{\mu_{\alpha_i}-1}x - E_{\mu_{\alpha_i}-2}\nu\left(\tilde{y}_i^{\mu_{\alpha_i}-1} - y_i^{\mu_{\alpha_i}-1}\right) \end{cases}$$

$1 \leq i \leq p$. Since (1) is BIBS and f, $\dot{f}$ are bounded, it can be shown (see [23] and [34]) that, with suitable gains in the output injections ν, a sliding mode appears in finite time on the manifolds $e_i^j = \dot{e}_i^j = 0$, $1 \leq i \leq p$, $1 \leq j \leq \mu_{\alpha_i} - 1$. Thus, the following equations hold after a finite time T:

$$\begin{matrix} \nu\left(y_i - y_i^1\right) = C_i Ax \\ \nu\left(\tilde{y}_i^2 - y_i^2\right) = C_i A^2 x \\ \vdots \\ \nu\left(\tilde{y}_i^{\mu_{\alpha_i}-1} - y_i^{\mu_{\alpha_i}-1}\right) = C_i A^{\mu_{\alpha_i}-1}x \end{matrix} \tag{14}$$

for $1 \leq i \leq p$, and

$$\tilde{y} \triangleq \begin{bmatrix} y_1 \\ \nu\left(y_1 - y_1^1\right) \\ \vdots \\ \nu\left(\tilde{y}_1^{\mu_{\alpha_1}-1} - y_1^{\mu_{\alpha_1}-1}\right) \\ \vdots \\ y_p \\ \vdots \\ \nu\left(\tilde{y}_p^{\mu_{\alpha_p}-1} - y_p^{\mu_{\alpha_p}-1}\right) \end{bmatrix} = \tilde{C}x. \tag{15}$$

4 The output feedback sliding mode control law

The triple $(A, B, \tilde{C})$ in (1) and (7) is now considered and a static output feedback controller developed.

4.1 Solution of the Existence Problem

This subsection will present a constructive analysis determining when and how the sliding surface parameter F can be constructed assuming the extended outputs are available. It is convenient to introduce, without loss of generality, a coordinate transformation to the usual regular form, making the final $\tilde{p}$ states of the system depend directly on the extended outputs [17]:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad \tilde{C} = \begin{bmatrix} 0 & T \end{bmatrix} \tag{16}$$

where $T \in \mathbb{R}^{\tilde{p}\times\tilde{p}}$ is an orthogonal matrix, $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and the remaining sub-blocks in the system matrix are partitioned accordingly. Define a corresponding switching surface parameter by $\tilde{F} \in \mathbb{R}^{m\times\tilde{p}}$. Let

$$\begin{array}{cc} \overset{\tilde{p}-m}{\leftrightarrow} & \overset{m}{\leftrightarrow} \end{array}$$
$$\begin{bmatrix} F_1 & F_2 \end{bmatrix} = \tilde{F}T$$

where T is the matrix from equation (16). As a result

$$\tilde{F}\tilde{C} = \begin{bmatrix} F_1 C_1 & F_2 \end{bmatrix}$$

where

$$C_1 = \begin{bmatrix} 0_{(\tilde{p}-m)\times(n-\tilde{p})} & I_{(\tilde{p}-m)} \end{bmatrix}$$

Therefore $\tilde{F}\tilde{C}B = F_2 B_2$ and the square matrix F_2 is nonsingular. The canonical form in (16) is a special case of the regular form normally used in sliding mode controller design, and the reduced-order sliding motion is governed by a free motion with system matrix

$$A_{11}^s = A_{11} - A_{12}F_2^{-1}F_1C_1 \tag{17}$$

which must therefore be stable. If $K \in \mathbb{R}^{m\times(\tilde{p}-m)}$ is defined as $K = F_2^{-1}F_1$ then

$$A_{11}^s = A_{11} - A_{12}KC_1$$

and the problem of hyperplane design is equivalent to a *static output feedback problem* for the system (A_{11}, A_{12}, C_1). In order to utilize the existing literature, it is necessary that the pair (A_{11}, A_{12}) is controllable and (A_{11}, C_1) is observable. The former is ensured as (A, B) is controllable. The observability of (A_{11}, C_1), is not so straightforward, but can be investigated by considering the canonical form below.

Lemma 2. *Let $(A, B, \tilde{C})$ be a linear system with $\tilde{p} > m$ and rank$(\tilde{C}B) = m$. Then a change of coordinates exists so that the system triple with respect to the new coordinates has the following structure:*

- *The system matrix can be written as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and the sub-block A_{11} when partitioned has the structure

$$A_{11} = \left[\begin{array}{cc|c} A_{11}^o & A_{12}^o & \multirow{2}{*}{A_{12}^m} \\ 0 & A_{22}^o & \\ \hline 0 & A_{21}^o & A_{22}^m \end{array}\right]$$

where $A_{11}^o \in \mathbb{R}^{r\times r}$, $A_{22}^o \in \mathbb{R}^{(n-\tilde{p}-r)\times(n-\tilde{p}-r)}$ and $A_{21}^o \in \mathbb{R}^{(\tilde{p}-m)\times(n-\tilde{p}-r)}$ for some $r \geq 0$ and the pair (A_{22}^o, A_{21}^o) is completely observable.
- *The input distribution matrix B and the output distribution matrix $\tilde{C}$ have the structure in (16).*

For a proof and a constructive algorithm to obtain this canonical form see [16].

In the case where $r > 0$, the intention is to construct a new system $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ which is both controllable and observable with the property that

$$\lambda(A_{11}^s) = \lambda(A_{11}^o) \cup \lambda(\tilde{A}_{11} - \tilde{B}_1K\tilde{C}_1).$$

As in [16], partition the matrices A_{12} and A_{12}^m as

$$A_{12} = \begin{bmatrix} A_{121} \\ A_{122} \end{bmatrix} \quad \text{and} \quad A_{12}^m = \begin{bmatrix} A_{121}^m \\ A_{122}^m \end{bmatrix}$$

where $A_{122} \in \mathbb{R}^{(n-m-r)\times m}$ and $A_{122}^m \in \mathbb{R}^{(n-\tilde{p}-r)\times(\tilde{p}-m)}$ and form a new subsystem $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$ where

$$\begin{aligned}\tilde{A}_{11} &= \begin{bmatrix} A_{22}^{o} & A_{122}^{m} \\ A_{21}^{o} & A_{22}^{m} \end{bmatrix} \\ \tilde{C}_1 &= \begin{bmatrix} 0_{(\tilde{p}-m)\times(n-\tilde{p}-r)} & I_{(\tilde{p}-m)} \end{bmatrix}\end{aligned} \tag{18}$$

It follows that the spectrum of A_{11}^{s} decomposes as

$$\lambda(A_{11} - A_{12}KC_1) = \lambda(A_{11}^{o}) \cup \lambda(\tilde{A}_{11} - A_{122}K\tilde{C}_1)$$

Lemma 3. *[16] The spectrum of A_{11}^{o} represents the invariant zeros of $(A, B, \tilde{C})$ which have been shown to be the invariant zeros of the original system triple (A, B, C).*

For a stable sliding motion, the invariant zeros of the system (A, B, C) must lie in the open left-half plane and the triple $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$ must be stabilizable with respect to output feedback. The matrix A_{122} is not necessarily full rank. Suppose $\text{rank}(A_{122}) = m'$ then, as in [16], it is possible to construct a matrix of elementary column operations $T_{m'} \in \mathbb{R}^{m\times m}$ such that

$$A_{122}T_{m'} = \begin{bmatrix} \tilde{B}_1 & 0 \end{bmatrix} \tag{19}$$

where $\tilde{B}_1 \in \mathbb{R}^{(n-m-r)\times m'}$ and is of full rank. If $K_{m'} = T_{m'}^{-1}K$ and $K_{m'}$ is partitioned compatibly as

$$K_{m'} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{matrix} \updownarrow m' \\ \updownarrow m-m' \end{matrix}$$

then

$$\begin{aligned}\tilde{A}_{11} - A_{122}K\tilde{C}_1 &= \tilde{A}_{11} - \begin{bmatrix} \tilde{B}_1 & 0 \end{bmatrix} K_{m'}\tilde{C}_1 \\ &= \tilde{A}_{11} - \tilde{B}_1K_1\tilde{C}_1\end{aligned}$$

and $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$ is stabilizable by output feedback if and only if $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ is stabilizable by output feedback. The triple must be controllable, observable and satisfy the Kimura–Davison conditions, which yield

$$m' + \tilde{p} + r \geq n + 1 \tag{20}$$

Lemma 4. *[16] The pair $(\tilde{A}_{11}, \tilde{B}_1)$ is completely controllable and $(\tilde{A}_{11}, \tilde{C}_1)$ is completely observable.*

The next subsection considers the problem of constructing an appropriate control law, based on the extended outputs only, to induce sliding.

4.2 The Reachability Problem

It will be shown that a sliding motion can be induced on the manifold $\{x \in \mathbb{R}^n : \tilde{F}\tilde{C}x = 0\}$ using a higher order sliding mode method (see [2], [24], or

[33] for further details). In order to stabilize the state of the system (1-2), the following output feedback controller is proposed:

$$u = \left(\tilde{F}\tilde{C}B\right)^{-1}\left(-\mathit{\Gamma}\tilde{F}\tilde{y} - w_{\tilde{y}}\right) \tag{21}$$

where $\mathit{\Gamma}$ is a strictly positive diagonal matrix and the auxiliary output $\tilde{y}$ is the output of the observer defined in §3.

$w_{\tilde{y}}$ is the super twisting algorithm defined componentwise by

$$\begin{cases} (w_{\tilde{y}})_i = \varphi\left((\tilde{F}\tilde{y})_i\right) + \lambda_i \left|(\tilde{F}\tilde{y})_i\right|^{\frac{1}{2}} \operatorname{sign}\left((\tilde{F}\tilde{y})_i\right) \\ \dot{\varphi}\left((\tilde{F}\tilde{y})_i\right) = \alpha_i \operatorname{sign}\left((\tilde{F}\tilde{y})_i\right) \end{cases}$$

where λ_i and α_i for $i = 1, ..., m$ are strictly positive constants. This algorithm has been developed for systems with relative degree 1 to avoid chattering phenomena. The control law is made of two continuous terms. The discontinuity only appears in the control input time derivative. Note that $\tilde{y}$ only depends on the output of the system and that the components of $\tilde{y}$ are smooth functions with discontinuous time derivatives.

The first time derivative of s_e is given by:

$$\begin{aligned} \dot{s}_e &= \tilde{F}\tilde{C}\left(Ax(t) + Bu(t) + f(t,x,u)\right) \\ &= \tilde{F}\tilde{C}Ax(t) + \tilde{F}\tilde{C}f(t,x,u) - \mathit{\Gamma}F\tilde{y} - w_{\tilde{y}} \end{aligned}$$

Because after a finite time T, $\tilde{y} = \tilde{C}x$, one has:

$$\dot{s}_e = \tilde{F}\tilde{C}Ax(t) + \tilde{F}\tilde{C}f(t,x,u) - \mathit{\Gamma}s_e - w_{\tilde{y}}$$

with

$$\begin{cases} (w_{\tilde{y}})_i = \varphi((s_e)_i) + \lambda_i \left|(s_e)_i\right|^{\frac{1}{2}} \operatorname{sign}((s_e)_i) \\ \dot{\varphi}((s_e)_i) = \alpha_i \operatorname{sign}((s_e)_i) \end{cases}$$

for $i = 1, ..., m$. The second time derivative s_e is given by:

$$\ddot{s}_e = \tilde{F}\tilde{C}A\left(Ax(t) + Bu(t) + f(\cdot)\right) + \tilde{F}\tilde{C}\dot{f}(\cdot) - \mathit{\Gamma}\dot{s}_e - \dot{w}_{\tilde{y}}$$

If the term $\left\|\tilde{F}\tilde{C}A\left(Ax(t) + Bu(t) + f(\cdot)\right) + \tilde{F}\tilde{C}\dot{f}(\cdot)\right\|$ is bounded and if the control gains satisfy the conditions given in [31], finite time convergence on the sliding surface $\{s_e = \dot{s}_e = 0\}$ is obtained. The system is asymptotically stabilized, because the sliding motion on $s_e = 0$ has a stable dynamics.

5 The sliding mode observer framework

The scheme described in this section will be based on a classical observer of the form (5) for the system $\{A, D, \tilde{C}\}$. Consequently this requires (in real-time) the outputs that correspond to $\tilde{C}x$ from knowledge of only $y = Cx$. In order to estimate the state of the system (1) with output (7), the following sliding mode observer is proposed:

$$\dot{z} = Az + Bu + G_l\left(\tilde{y} - \tilde{C}z\right) + G_n v_c \tag{22}$$

The discontinuous output injection v_c from (22) is defined by:

$$v_c = \begin{cases} \rho \frac{P_2(\tilde{y}-\tilde{C}z)}{\|P_2(\tilde{y}-\tilde{C}z)\|} & \text{if } (\tilde{y} - \tilde{C}z) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

where ρ is a positive constant larger than the upper bound of w. The definition of the symmetric positive definite matrix P_2 can be found in [15] or in Chapter 6 of [17]. Still, $\tilde{y}$ is the output of the observer defined defined in §3.

Define the state estimation error $e = x - z$. Then it is straightforward to show that:

$$\dot{e} = Ae + D\xi - G_l\left(\tilde{y} - \tilde{C}z\right) - G_n v_c \tag{24}$$

After a finite time T, $\tilde{y} = \tilde{C}x$ and for all $t > T$, the error dynamics (24) are given by:

$$\dot{e} = \left(A - G_l\tilde{C}\right)e + D\xi - G_n v_c \tag{25}$$

with

$$v_c = \begin{cases} \rho \frac{P_2(\tilde{C}e)}{\|P_2(\tilde{C}e)\|} & \text{if } (\tilde{C}e) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

Since by construction rank$(\tilde{C}D) =$ rank(D) and by assumption the invariant zeros of the triple $(A, D, \tilde{C})$ lie in the left half plane, the design methodologies given in [15], [17] or [38] can be applied so that $e = 0$ is an asymptotically stable equilibrium point of (25) and the dynamics are independent of ξ once a sliding motion on the sliding manifold $\left\{e : s = \tilde{C}e = 0\right\}$ has been attained.

In addition, the method enables estimation of the unknown inputs. Define $(v_c)_{eq}$ as the equivalent output error injection required to maintain the sliding motion in (25). During the sliding motion,

$$\dot{s} = \tilde{C}\,\dot{e} = \tilde{C}\left(A - G_l\tilde{C}\right)e + \tilde{C}D\xi - \tilde{C}G_n v_c\left(\tilde{C}e\right) = 0$$

Since $e \to 0$ and using (25):

$$\tilde{C}G_n\left(v_c\right)_{eq} \to \tilde{C}D\xi.$$

As $\tilde{C}D$ is full rank, an approximation $\hat{\xi}$ of ξ can be obtained from $(v_c)_{eq}$ by:

$$\hat{\xi} = \left(\left(\tilde{C}D\right)^T\tilde{C}D\right)^{-1}\left(\tilde{C}D\right)^T\tilde{C}G_n\left(v_c\right)_{eq}.$$

Note that during the sliding motion, the effect of the linear feedback $G_l \tilde{C} e$ disappears since $\tilde{C} e \equiv 0$. However the inclusion of G_l is involved with the proof of global convergence of e to zero and the construction of a Lyapunov function [15]. It also means that prior to sliding, (24) can be viewed as having filtering properties since by construction $A - G_l C$ is stable.

Remark 1. There exist in the literature several sliding mode observers that do not require the relative degree condition (in [1, 12, 13, 23, 25, 31]). All these works deal with finite time state estimation and only consider systems without invariant zeros. Except for [23] and [31], these papers do not consider unknown inputs affecting the system and focus on the estimation of the states.

6 Speed and tension control in a web-transport system

The control of winding systems for handling webbed material such as textiles, paper, polymers and metals is of great industrial interest. Independent control of the velocity and tension of the material in the face of time-varying parameter changes in the radius of the material on the winder and unwinder reels is required. A model of a three motor winding system is given in [3] as described below:

$$N\dot{x} = Ax + Bu \tag{27}$$

$$y = Cx \tag{28}$$

where

$$x = [V_1, T_1, V_2, T_2, V_3, T_3, V_4, T_4, V_5]^T$$

and $u = [u_u, u_v, u_w]^T$ and $y = [T_u, V_3, T_w]^T$. The control inputs are the torque control signals applied to three brushless motors driving the unwinder, the master tractor and the winder respectively. The output measurements are the web tensions at the unwinder and winder, T_u and T_w, respectively, and the web velocity, V_3, measured at the master tractor. The states of the system are the corresponding tensions, T_i, and web velocities V_i at various points across the process. The matrices A, B, C and N are given below:

$$A = \begin{pmatrix} -f_1 & R_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -E_0 & -V_0 & E_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_2^2 & -f_2 & R_2^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & V_0 & -E_0 & -V_0 & E_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_3^2 & -f_3 & R_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & V_0 & -E_0 & -V_0 & E_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -R_4^2 & E_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V_0 & -E_0 & -V_0 & E_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_5^2 & -f_5 \end{pmatrix} \tag{29}$$

$$B = \begin{pmatrix} K_u R_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & K_t R_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_w R_5 \end{pmatrix} \tag{30}$$

$$C = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \end{pmatrix} \tag{31}$$

$$N = \mathrm{diag}\{J_1, L, J_2, L, J_3, L, J_4, L, J_5\} \tag{32}$$

In the above matrices, V_i, R_i, J_i and f_i are the linear velocity, the radius, the inertia and the viscous friction coefficient of the ith roll, L is the web length between two successive rolls, and K_u, K_t and K_w are the torque constants of the three motors. V_0 and E_0 are the nominal values of the linear web velocity and the elastic modulus of the material respectively. The nominal data values used to construct a linear model at start-up are taken from [29] and reported in Table 1.

Notation	Value	Units	Notation	Value	Units
L	0.45	m	J_2	0.00109	kg.m^2
V_0	100/60	m.s^{-1}	J_3	0.00184	kg.m^2
E_0	4175	N.m	J_4	0.00109	kg.m^2
R_1	0.031	m	J_5	0.00109	kg.m^2
R_2	0.02	m	f_1	0.0195	N.m.s.rad^{-1}
R_3	0.035	m	f_2	0.000137	N.m.s.rad^{-1}
R_4	0.02	m	f_3	0.0075	N.m.s.rad^{-1}
R_5	0.032	m	f_4	0.000466	N.m.s.rad^{-1}
J_1	0.0083	kg.m^2	f_5	0.0045	N.m.s.rad^{-1}

Table 1. Parameters of the winding machine

Consider first the problem of output feedback control. For simplicity and without loss of generality, it is assumed that the system (27) is not subject to disturbances f. Note that $\det(CN^{-1}B) = 0$ and so standard approaches for the design of sliding mode controllers based on output measurements cannot be applied to this system. The nominal system also possesses four transmission zeros located at $-3.7500 \pm 86.0751i$ and $-2.1500 \pm 96.3089i$. The winding system is thus seen to represent an appropriate case study to illustrate the work presented in this chapter. Define:

$$\tilde{C} = \begin{bmatrix} C_1 \\ C_1 N^{-1} A \\ C_2 \\ C_3 \\ C_3 N^{-1} A \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{E_0}{2L} & 0 & 0 & -\frac{V_0}{2L} & \frac{E_0}{2L} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{V_0}{2L} & -\frac{E_0}{2L} & 0 & 0 & -\frac{V_0}{2L} & \frac{E_0}{2L} \end{bmatrix}. \tag{33}$$

Then it is easy to verify $\tilde{C}B$ has full rank. Thus, the extended output is given by

$$\tilde{y} = \begin{bmatrix} y_1 \\ \nu\left(y_1 - \hat{y}_1\right) \\ y_2 \\ y_3 \\ \nu\left(y_3 - \hat{y}_3\right) \end{bmatrix}. \tag{34}$$

and second and fifth outputs in $\tilde{y}$ are produced from (in this case) the degenerate step-by-step observers

$$\dot{\hat{y}}_1 = \nu\left(y_1 - \hat{y}_1\right) + C_1 N^{-1} B u$$
$$\dot{\hat{y}}_3 = \nu\left(y_3 - \hat{y}_3\right) + C_3 N^{-1} B u$$

where ν is defined by (11).

As tracking of desired values for the web tensions at the unwinder and winder, T_u and T_w, respectively, and the web velocity, V_3, is desired, an integral action methodology is employed. Define additional integral action states as

$$\dot{\eta} = -Cx + \eta y_{ref} \tag{35}$$

Note that the integral action states are defined for the measured plant outputs, and not the extended output. For the design of the sliding surface, the output matrix is defined in terms of the augmented state $[x, \eta]^T$ and given by

$$\begin{bmatrix} \tilde{C} & 0 \\ 0 & I_3 \end{bmatrix} \tag{36}$$

With this augmented output, the introduction of integral action produces no additional transmission zeros and the design procedure follows that described earlier. With 12 states, 3 inputs and 8 effective outputs, the reduced order stabilization problem defining the switching surface dynamics is a state feedback problem and the 5 poles available for selection are placed at -0.75, -2, -0.5, -1 and -1.5. These poles, together with the transmission zeros of the original system, wholly determine the dynamics in the sliding mode. The extended outputs are formed by two differentiators, each with gains $l_1 = 1000$ and $l_2 = 500$. The parameters of the twisting algorithm are given by $\alpha_1 = 2$, $\lambda_1 = 1$, $\alpha_2 = 200$, $\lambda_2 = 100$, $\alpha_3 = 200$ and $\lambda_3 = 50$. Under nominal operating conditions, where fixed parameter values are used and no nonlinearity is present, the system was required to track a velocity offset command of 1 m/s and desired offset tensions of 0.1N. After 20 seconds, a further offset of 0.5N

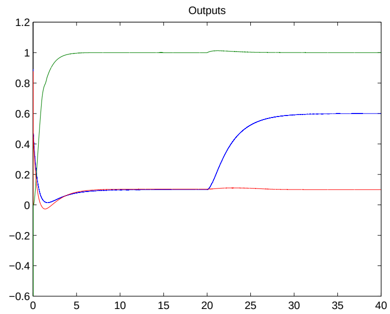


Fig. 1. Evolution of the outputs for the nominal linear plant model

was demanded at the winder. As can be seen from Figure 1, the set point changes are accurately attained with minimal coupling between the outputs.

In Figure 2, the evolution of the outputs is shown for the same reference demands, but this time in the presence of variations in the radii of the rolls. The system starts with a diameter of 0.15 m of material on the unwinder. During the nonlinear simulation, the material is wound onto the winder. The simulation incorporates parameter changes and the nonlinear behaviour of the radii. Visually, the results are seen to be very similar. The robustness of the method is thus highlighted.

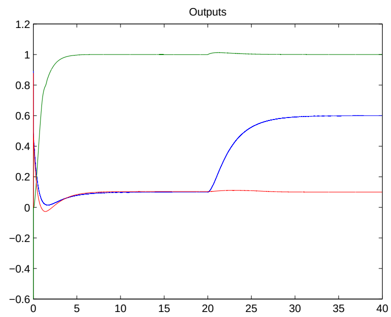


Fig. 2. Evolution of the outputs for the nonlinear, time varying plant

For the observer design problem, consider the same three motor winding systems with unknown inputs:

$$N\dot{x} = Ax + Bu + D\xi \tag{37}$$

$$y = Cx \tag{38}$$

The signal $\xi = \left[\, \xi_1(t)\ \xi_2(t)\ \xi_3(t) \,\right]^T$ represents the unknown inputs vector and it is assumed the unknown input distribution matrix is given by $D = B$. Thus, the bounded signal ξ_i may represent an actuator fault in such a way that $\xi_i(t) \neq 0$ when a fault appears and is zero in the fault free case. Since $det(CN^{-1}D) = 0$, standard sliding mode observer and other UIO approaches cannot be applied to this system but the augmented output used for the control problem may be employed. Consequently using the ideas in §5, the following 'classical' sliding mode observer can be designed:

$$\dot{z} = N^{-1}Az + N^{-1}Bu + G_l\left(\tilde{y} - \tilde{C}z\right) + G_n v_c$$

where v_c is the discontinuous (unit vector) output injection term as in (23), and $\tilde{y}$ and $\tilde{C}$ are given by (34) and (33), respectively. Define the observation errors as $e = x - z$ and $e_{y_1} = y_1 - \hat{y}_1$, $e_{y_3} = y_3 - \hat{y}_3$. Then the error dynamics are given by:

$$\dot{e} = N^{-1}Ae + N^{-1}D\xi - G_l\left(\tilde{y} - \tilde{C}z\right) - G_n v_c \tag{39}$$

$$\begin{aligned} \dot{e}_{y_1} &= C_1\left(N^{-1}Ax + N^{-1}Bu + N^{-1}D\xi\right) - \nu\left(e_{y_1}\right) - C_1N^{-1}Bu \\ &= C_1N^{-1}Ax - \nu\left(e_{y_1}\right) \end{aligned} \tag{40}$$

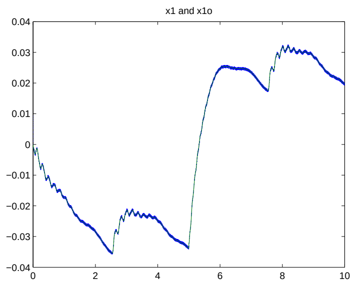
$$\begin{aligned} \dot{e}_{y_3} &= C_3\left(N^{-1}Ax + N^{-1}Bu + N^{-1}D\xi\right) - \nu\left(e_{y_3}\right) - C_3N^{-1}Bu \\ &= C_3N^{-1}Ax - \nu\left(e_{y_3}\right) \end{aligned} \tag{41}$$

As in [31], choose λ_s and α_s large enough such that after a finite time T_i, $e_{y_i} = \dot{e}_{y_i} = 0$, and $\nu\left(e_{y_i}\right) = C_iN^{-1}Ax$, $i = 1, 3$. This implies that for $t > \max\{T_1, T_3\}$, system (39)-(41) becomes:

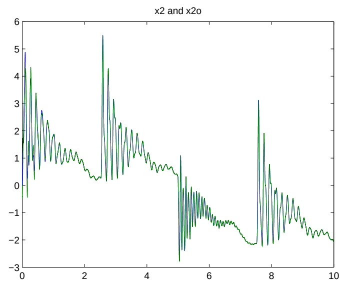
$$\begin{aligned} \dot{e} &= \left(N^{-1}A - G_l\tilde{C}\right)e + N^{-1}D\xi - G_n v_c \\ \dot{e}_{y_1} &= \dot{e}_{y_3} = 0 \end{aligned}$$

In the simulations, the following observer parameters have been chosen. The two scalar gains associated with the observers to estimate $\dot{y}_1$ and $\dot{y}_3$ are $\lambda_s = 300$ and $\alpha_s = 8000$. The scalar gain associated with the first order sliding mode discontinuous injection v_c is $\rho = 1.5$. The control signal u has been set to zero without loss of generality. The unknown inputs have been chosen as follows: ξ_1 is a square wave of amplitude 0.1 and frequency $0.1Hz$ that starts at $t = 5s$; ξ_2 is a sine wave of amplitude 0.2 and frequency $1Hz$ that starts at $t = 0s$; ξ_3 is a sawtooth signal of amplitude 0.05 and frequency $0.4Hz$ that starts at $t = 0s$. Note that at a finite set of isolated points in time, the derivatives of the unknown input signals do not exist. However the differential equations are satisfied almost everywhere. Figures 3 and 4 show that the state is accurately estimated in spite of the three actuator faults. It can be seen in Figure 5 that the unknown input signals are also accurately reconstructed.

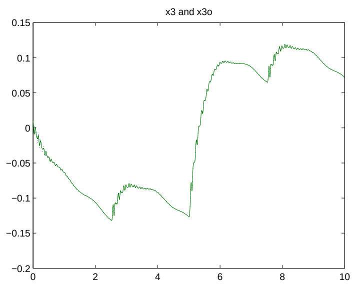
A simulation has been made with a 10% variation of the viscous coefficient f_2. Again, all states were recovered as well as the three unknown inputs. This is shown in Figure 6. Another simulation for testing robustness issue has been realized by considering a 20% variation of Young modulus E_0. The results of the unknown input reconstruction are shown in Figure 7. The numerical results indicate that the actuator fault detection scheme is tractable even with parameter uncertainties. This is important for instance if several materials with different Youngs modulus have to be used on the same winding machine.



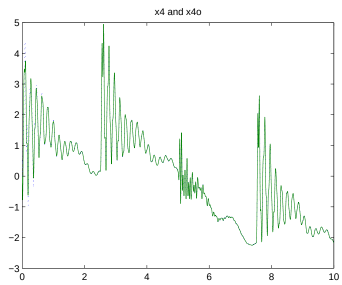
(a)



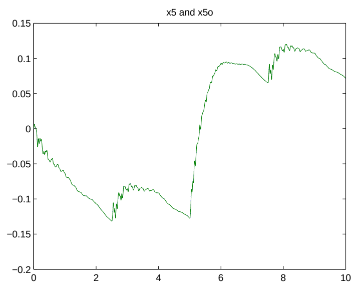
(b)



(c)



(d)



(e)

Fig. 3. State and estimation (nominal case)

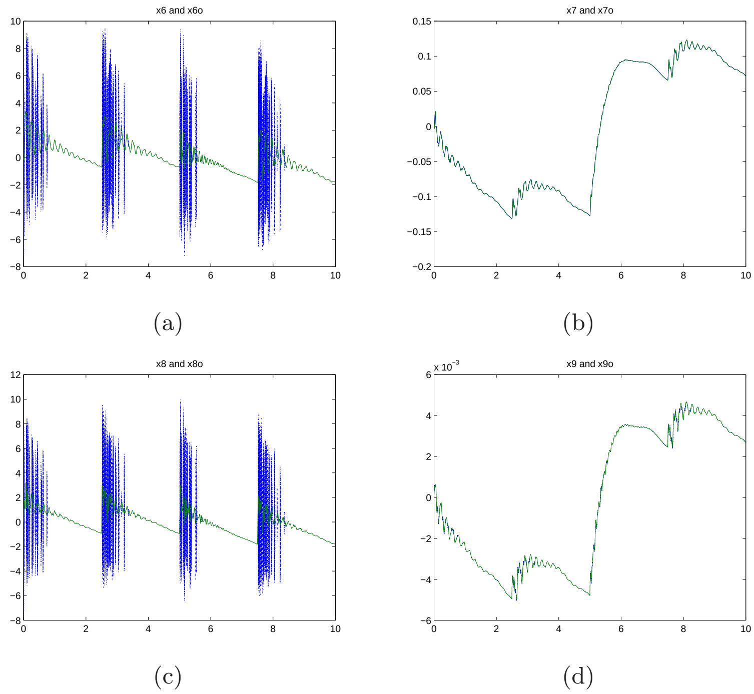


Fig. 4. State and estimation (nominal case)

7 Concluding remarks

This chapter has presented both an output feedback sliding mode control design framework and a sliding mode observer design approach for MIMO systems of arbitrary relative degree. A minimal set of outputs and output derivatives have been identified to determine an augmented system which is relative degree one, and a robust sliding mode differentiator has been used as a mechanism to construct the extended output signal. It has been shown that the transmission zeros of the original triple appear directly in the reduced order sliding mode dynamics relating to the augmented system. For the static output feedback control problem a super twisting control algorithm has been shown to provide robust control performance. The problem of designing a sliding mode unknown input observer for linear systems has been broadened using the same approach. The scheme is based on a 'classical' sliding mode observer used in conjunction with a scheme to estimate a certain number of derivatives of the outputs. The number of derivatives required is system dependent and can be easily calculated. By using the equivalent output injections from the derivative estimation scheme and the classical observer, estimation of both

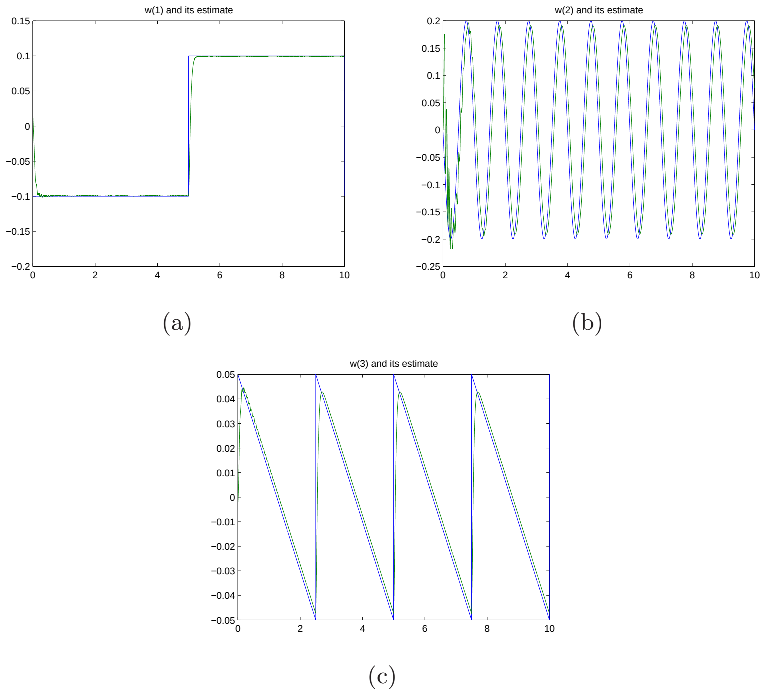


Fig. 5. Unknown input and estimation (nominal case)

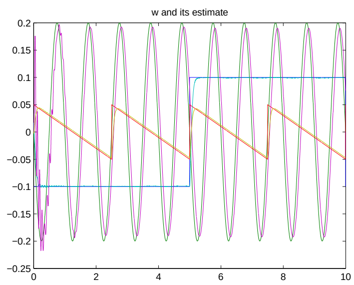


Fig. 6. Unknown input and estimation: 10% variation of the viscous coefficient f_2

the system state and the unknown inputs can be obtained. Since the derivative estimation observer is based on second order sliding mode algorithms, the equivalent output injections are obtained in a continuous way without the use of low pass filters.

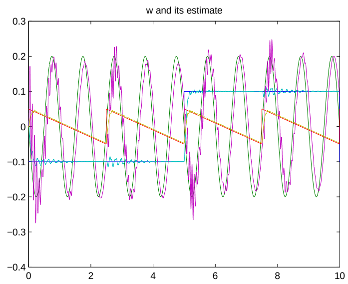


Fig. 7. Unknown input and estimation: 20% variation of Young modulus E_0

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